NOISE FIGURE IN DIGITAL SYSTEMS

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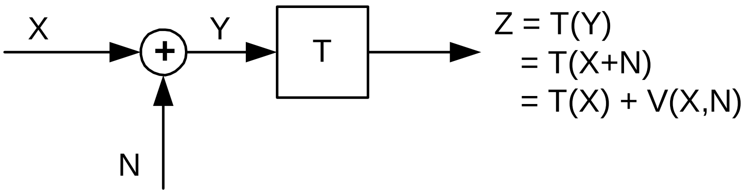
1. Introduction

In a digital system we deal with signals affected by quantization noise which are transformed through numerical processing in other signals to obtain the required result of the algorithm we are implementing. Each transformation creates a new signal Z modifying both the original signal X and the associated quantization noise N. The Z signal is composed by two parts: the wanted signal T(X) and its associated quantization noise V(X,N), depending on both X and N, and has its own SNR.



A noise figure of the transformation T can be defined as:

$$NF(T) = \frac{SNR(Y)}{SNR(Z)} \qquad \text{with:} \qquad SNR(Y) = \frac{P(X)}{P(N)}, \qquad SNR(Z) = \frac{P(T(X))}{P(V(X,N))}$$

Where P(A) means “power of A”.
In many cases a first order approximation of Z is enough to achieve a correct analysis of NF(T). Using the McLaurin expansion of Z with respect to the variable N we can write:

$$Z = T(X+N) \approx T(X) + T_N(X)\,N \qquad \text{being:} \qquad T_N(X) = \left(\frac{\partial T(X+N)}{\partial N}\right)_{N=0}$$

If we consider the signals X and N as independent random variables NF(T) can then be written as:

$$NF(T) = \frac{SNR(Y)}{SNR(Z)} = \frac{P(X)}{P(N)} \frac{P(T_N(X))\,P(N)}{P(T(X))} = P(T_N(X)) \frac{P(X)}{P(T(X))}$$

which does not dependent on N.
If X and N have uniform distribution between $[-x_1 \ldots x_2]$ and $[-n \ldots n]$ respectively, we can achieve a model which allows the explicit computation of NF(T) for many relevant T.

2. Transformation of PDF

It is possible to use the following theorem to obtain the probability density function of T(X).

Theorem 1

Let X be a continuous random variable and T a slightly monotonic function on the domain spanned by X. Defined Z=T(X), the PDF (Probability Density Function) of Z is given by:

$$p_Z(z) = p_X(T^{-1}(z)) \left| \frac{dT^{-1}(z)}{dz} \right|$$

Proof A. If T is monotonic increasing, the CDF (Cumulative Distribution Function) $F_Z(z)$ of Z=T(X) is given by:

$$F_Z(z) = \Pr[Z \leq z] = \Pr[T(X) \leq T(x)] = \Pr[X \leq x] = F_X(x)$$

The PDF of Z maybe found as follow:

$$p_Z(z) = \frac{dF_Z(z)}{dz} = \frac{dF_X(x)}{dz} = \frac{dF_X(x)}{dx}\frac{dx}{dz} = p_X(T^{-1}(z))\frac{dT^{-1}(z)}{dz} = p_X(T^{-1}(z)) \left| \frac{dT^{-1}(z)}{dz} \right|$$

Being T and T^{-1} increasing functions (derivative > 0). Analogously if T is monotonic decreasing, the CDF is given by:

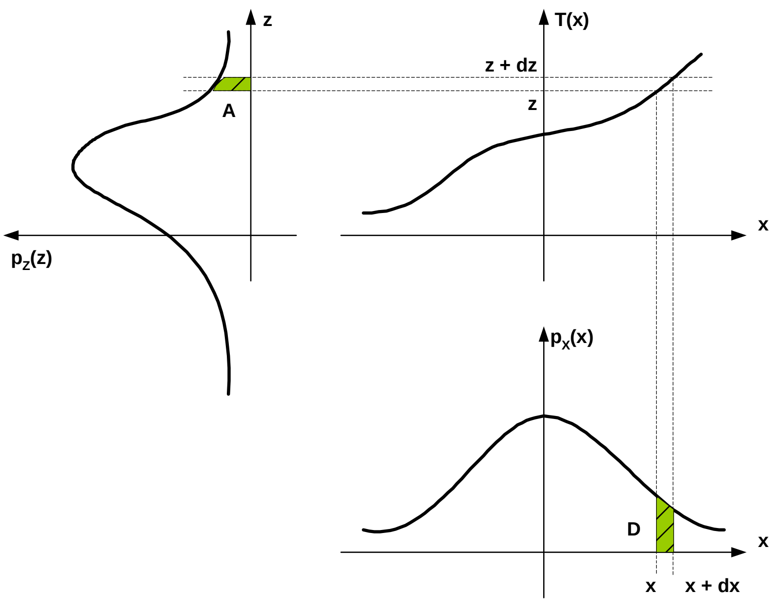
$$F_Z(z) = \Pr[Z \leq z] = \Pr[T(X) \leq T(x)] = \Pr[X > x] = 1 - \Pr[X \leq x] = 1 - F_X(x)$$

So the PDF is:

$$p_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d(1-F_X(x))}{dz} = -\frac{dF_X(x)}{dx}\frac{dx}{dz} = -p_X(T^{-1}(z))\frac{dT^{-1}(z)}{dz} = p_X(T^{-1}(z)) \left| \frac{dT^{-1}(z)}{dz} \right|$$

Being T and T^{-1} decreasing functions (derivative < 0). Considering both the results we have the thesis.
End Proof A.

Proof B. The theorem can also be proven as follow. After the increasing transformation T we obtain a random variable with a different PDF but same CDF: A=D in the figure below.



Then:

$$A = D \quad \rightarrow \quad p_Z(z)\,dz = p_X(x)\,dx \quad \rightarrow \quad p_Z(z) = p_X(x)\frac{dx}{dz} = p_X(T^{-1}(z))\frac{dT^{-1}(z)}{dz}$$

For a decreasing transformation we would obtain a sign change in both the second member of the previous formula and in its derivative being, as in the previous proof, $dT^{-1}(z)/dz<0$:

$$A = -D \quad \rightarrow \quad p_Z(z)\,dz = -p_X(x)\,dx \quad \rightarrow \quad p_Z(z) = p_X(x)\frac{dx}{dz} = -p_X(T^{-1}(z))\frac{dT^{-1}(z)}{dz}$$

So, the formula including both cases is:

$$p_Z(z) = p_X(T^{-1}(z))\left|\frac{dT^{-1}(z)}{dz}\right|$$

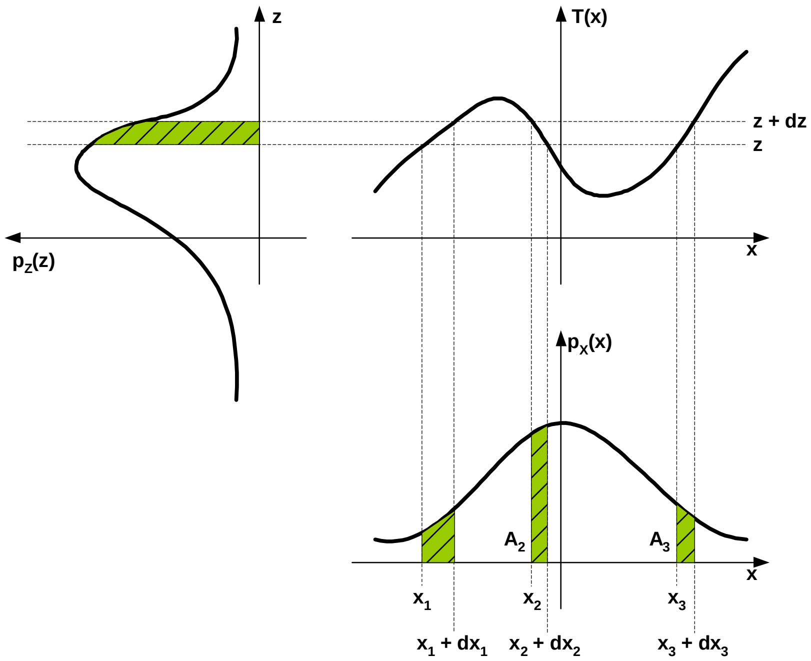
End Proof B.

Theorem 2

Let X be a continuous random variable and T a non-monotonic function which can be split into a finite number of monotonic sections on the domain spanned by X. Defined Z=T(X), the PDF (Probability Density Function) of Z is:

$$p_Z(z) = \sum_{k=1}^{q} \left(p_X(T^{-1}(z)) \right)_{T^{-1}(z) = x_k} \left| \frac{d}{dz}\left((T^{-1}(z))_{T^{-1}(z) = x_k} \right) \right|$$

Being x_k the q roots of the equation $T^{-1}(z) = x$, that is $T(x) = z$, for any given z. This equation has more than one solution because the function T is no more monotonic as shown in figure:



Proof. Considering the figure above, we can give a simple proof for the case in which only three intersections are present (the generalization to any number of intersections is straightforward). Since must be $A=A_1+A_2+A_3$ we have:

$$p_Z(z)\,dz = p_X(x_1)\,dx_1 + p_X(x_2)\,dx_2 + p_X(x_3)\,dx_3 \qquad \text{so:} \qquad p_Z(z) = \sum_{k=1}^{3}\left(p_X(x_k)\frac{d\,x_k}{dz} \right)$$

End Proof.

Example 1: PDF of $T(X) = X^2$

T is not monotonic but formed by two monotonic sections. Given z we have two different solution for the equation T(x)=z: $x=\pm\sqrt{z}$ so also: $T^{-1}(z)=x=\pm\sqrt{z}$. Using the previous Theorem 2 then:

$$p_Z(z) = p_X(\sqrt{z}) \left| \frac{1}{2\sqrt{z}} \right| + p_X(-\sqrt{z}) \left| \frac{1}{-2\sqrt{z}} \right| = \left(p_X(\sqrt{z}) + p_X(-\sqrt{z}) \right) \frac{1}{2\sqrt{z}}$$

If for example $p_X(x)$ is uniformly distributed in $[-x_1, x_2]$, with $0 < x_1 < x_2$, and zero elsewhere, then we have:

$$p_Z(z) = \begin{cases} \dfrac{1}{x_2 + x_1} \dfrac{1}{\sqrt{z}} & -x_1 \le x \le x_1 \\ \dfrac{1}{x_2 + x_1} \dfrac{1}{2\sqrt{z}} & (-x_2 \le x < -x_1) \cup (x_1 < x \le x_2) \\ 0 & x_2 < x < -x_2 \end{cases}$$

If $x_1=x_2=a$, the previous formula simplifies as:

$$p_Z(z) = \begin{cases} \dfrac{1}{2a} \dfrac{1}{\sqrt{z}} & -a \le x \le a \quad \{ 0 \le z \le a^2 \} \\ 0 & a < x < -a \quad \{ a^2 \le z \} \end{cases}$$

Example 2: PDF of T(X) = SIN(X)

T is not monotonic but formed by three monotonic sections when we consider $0 \le X \le 2\pi$: increasing from 0 to $\pi/2$ and from $3\pi/4$ to 2π, decreasing from $\pi/2$ to $3\pi/4$. Being asin defined in $[-\pi/2 \ldots \pi/2]$, given z we have two different solution for the equation T(x)=z in $[0 \ldots 2\pi]$:

1) $$x_1 = \begin{cases} \operatorname{asin}(z) & z \ge 0 \\ \operatorname{asin}(z) + 2\pi & z < 0 \end{cases} = \operatorname{asin}(z) \bmod 2\pi$$

2) $$x_2 = \pi - \operatorname{asin}(z)$$

Using the previous Theorem 2 we have:

$$p_Z(z) = p_X(x_1) \left| \frac{d}{dz} (\operatorname{asin}(z) \bmod 2\pi) \right| + p_X(x_2) \left| \frac{d}{dz} (\pi - \operatorname{asin}(z)) \right| = \left(p_X(x_1) + p_X(x_2) \right) \frac{1}{\sqrt{1 - z^2}}$$

If $p_X(x)$ is uniformly distributed in $[0 \ldots 2\pi]$ and zero elsewhere, then we have:

$$p_Z(z) = \begin{cases} \dfrac{2}{2\pi} \dfrac{1}{\sqrt{1 - z^2}} & 0 \le x \le 2\pi \\ 0 & 2\pi < x < 0 \end{cases}$$

Example 3: PDF of T(X) = COS(X)

T is not monotonic but formed by two monotonic sections when we consider $0 \le X \le 2\pi$: decreasing from 0 to π, increasing from π to 2π. Being acos defined in $[0 \ldots \pi]$, given z we have two different solution for the equation T(x)=z in $[0 \ldots 2\pi]$:

1) $x_1 = \mathrm{acos}(z)$

2) $x_2 = 2\pi - \mathrm{acos}(z)$

Using the previous Theorem 2 we have:

$$p_Z(z) = p_X(x_1)\left|\frac{d}{dz}(\mathrm{acos}(z))\right| + p_X(x_2)\left|\frac{d}{dz}(2\pi - \mathrm{acos}(z))\right| = \left(p_X(x_1) + p_X(x_2)\right)\frac{1}{\sqrt{1-z^2}}$$

If $p_X(x)$ is uniformly distributed in $[0\ldots2\pi]$ and zero elsewhere, then we have:

$$p_Z(z) = \begin{cases} \frac{2}{2\pi}\frac{1}{\sqrt{1-z^2}} & 0 \le x \le 2\pi \\ 0 & 2\pi < x < 0 \end{cases}$$

that is the same obtained for the sin function.

Example 4: PDF of $T(X) = \sqrt{X}$

T is monotonic for $0 \le X \le a$ and $T^{-1}(z) = z^2$ then, using Theorem 1:

$$p_Z(z) = p_X(z^2)\,|\,2\,z\,|$$

if X is uniformly distributed in [0..a] and zero elsewhere:

$$p_Z(z) = \frac{2\,z}{a}$$

3. Power of Random Variables

Definition 1: Power of a Random Variable

The power of a random variable is:

$$P(X) = \int_{-\infty}^{+\infty} x^2\, p_X(x)\, dx$$

End Definition.

The power of Z=T(X) can be put then, according to the Theorem 2, in the form:

$$P(T(X)) = \int_{-\infty}^{+\infty} z^2\, p_Z(z)\, dz = \int_{-\infty}^{+\infty} z^2 \sum_{k=1}^{q} \left(p_X(T^{-1}(z)) \left| \frac{d\,T^{-1}(z)}{dz} \right| \right)_{T^{-1}(z) = x_k} dz$$

Being x_k the q roots of the equation $T^{-1}(z) = x$.
These formulas can be simplified when T is monotonic and X has a uniform distribution between $(x_1 \ldots x_2)$. In that case:

$$P(X) = \int_{x_1}^{x_2} x^2 \frac{1}{x_2 - x_1}\, dx = \frac{1}{x_2 - x_1} \left(\frac{x_2^3}{3} - \frac{x_1^3}{3} \right) = \frac{1}{3} \frac{x_2^3 - x_1^3}{x_2 - x_1} \qquad \text{and:}$$

$$p_Z(z) = \begin{cases} \dfrac{1}{x_2 - x_1} \left| \dfrac{dT^{-1}(z)}{dz} \right| & x_1 \le x \le x_2 \\ 0 & x_2 \le x \le x_1 \end{cases} \quad \rightarrow \quad P(T(X)) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} z^2 \left| \frac{dT^{-1}(z)}{dz} \right| dz$$

Example 5: Power of T(X) = X²

Assuming X uniformly distributed in [-a…a] and using the PDF of X^2 already computed in the previous Example 1:

$$P(X^2) = \int_{-\infty}^{+\infty} z^2\, p_z(z)\, dz = \frac{1}{2a} \int_{0}^{a^2} z^2 \frac{1}{\sqrt{z}}\, dz = \frac{1}{5a} \left[z^{5/2} \right]_{z=0..a^2} = \frac{1}{5a} \sqrt{a^{10}} = \frac{a^4}{5}$$

Whereas the power of X is:

$$P(X) = \int_{-\infty}^{+\infty} x^2\, p_x(x)\, dz = \frac{1}{2a} \int_{-a}^{a} x^2\, dx = \frac{1}{2a} \left[\frac{x^3}{3} \right]_{z=-a..a} = \frac{1}{2a} \frac{2a^3}{3} = \frac{a^2}{3}$$

Example 6: Power of T(X) = SIN(X) and COS(X)

Since they have the same distribution (see Example 2 and Example 3), the transformed variables have the same power: P(SIN(X))=P(COS(X)).

Example 7: Power of T(X) = √ X

Assuming X uniformly distributed in [0…a] and using the PDF of √X already computed in the previous Example 4:

$$P(X^2) = \int_{-\infty}^{+\infty} z^2\, p_z(z)\; dz = \frac{1}{2a} \int_{0}^{\sqrt{a}} z^2 \frac{2z}{a}\; dz = \frac{2}{a}\left[\frac{z^4}{4}\right]_{z=0..\sqrt{a}} = \frac{2}{a}\frac{a^2}{4} = \frac{a}{2}$$

Whereas the power of X is:

$$P(X) = \int_{-\infty}^{+\infty} x^2\, p_x(x)\; dz = \frac{1}{a}\int_{0}^{a} x^2\; dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_{z=0..a} = \frac{1}{a}\frac{a^3}{3} = \frac{a^2}{3}$$

4. Transformations

4.1. The SIN Transformation

Letting T=SIN with $(x_1 \ldots x_2)=(0 \ldots 2\pi)$ and $n \ll 2\pi$ we have:

$$Z = SIN(X+N) = SIN(X)\,COS(N) + COS(X)\,SIN(N) \approx SIN(X) + N\,COS(X) \quad , \quad \text{so:}$$

$$SNR(Z) = \frac{P(SIN(X))}{P(N\,COS(X))} = \frac{1}{P(N)}$$

since N and X are independent and P(SIN(X))=P(COS(X)). Considering SNR(Y)=P(X)/P(N) we obtain then:

$$NF(T) = \frac{SNR(Y)}{SNR(Z)} = P(X)$$

The same result can be achieved using the first order approximation for Z:

$$T_N(X) = \left(\frac{\partial\, SIN(X+N)}{\partial N} \right)_{N=0} = \left(COS(X+N) \right)_{N=0} = COS(X)$$

and:

$$NF(T) = P(T_N(X))\,\frac{P(X)}{P(T(X))} = P(COS(X))\,\frac{P(X)}{P(SIN(X))} = P(X)$$

being P(SIN(X))=P(COS(X)).
Finally, considering the already computed P(X) we have:

$$NF(T) = P(X) = \frac{1}{3}\,\frac{x_2^3 - x_1^3}{x_2 - x_1} = \frac{4\pi^2}{3} \quad \rightarrow \quad NF(T)_{dB} = 10 \log \frac{4\pi^2}{3} = 11.192 \text{ dB}$$

4.2. The SQUARE Transformation

Letting $T=()^2$ with $(x_1 \ldots x_2)=(-a \ldots a)$, then:

$$Z = (X+N)^2 = X^2 + 2XN + N^2 \qquad \text{so:}$$

$$SNR(Z) = \frac{P(X^2)}{P(N\,(2X+N)\,)} = \frac{P(X^2)}{P(N)\,[P(2X) + P(N)]} = \frac{P(X^2)}{P(N)\,[4P(X) + P(N)]}$$

because N and X are independent. Considering SNR(Y)=P(X)/P(N) we obtain then:

$$NF(T) = \frac{SNR(Y)}{SNR(Z)} = \frac{P(X)\,[4P(X) + P(N)]}{P(X^2)} \approx \frac{4\,P^2(X)}{P(X^2)}$$

Having used the usually true condition $P(N) \ll P(X)$.
The same result can be achieved using the first order approximation for Z:

$$T_N(X) = \left(\frac{\partial\,(X+N)^2}{\partial N} \right)_{N=0} = \left(2(X+N) \right)_{N=0} = 2X$$

and:

$$NF(T) = P(T_N(X))\,\frac{P(X)}{P(T(X))} = 4\,P(X)\,\frac{P(X)}{P(X^2)} = \frac{4\,P^2(X)}{P(X^2)}$$

When X is uniformly distributed in [-a…a], using the results of the Example 5:

$$NF(T) \approx \frac{4\,P^2(X)}{P(X^2)} = 4\,\frac{a^4}{9}\,\frac{5}{a^4} = \frac{20}{9} \quad \rightarrow \quad NF(T)_{dB} = 10\log\frac{20}{9} = 3.468\text{ dB}$$

4.3. The SQUARE ROOT Transformation

Letting T=√ with $(x_1 \ldots x_2)=(0 \ldots a)$, then:

$$Z = \sqrt{X+N} \approx \sqrt{X} + T_N(X)\,N \qquad \text{where:} \quad T_N(X) = \left(\frac{\partial \sqrt{X+N}}{\partial N} \right)_{N=0} = \left(\frac{1}{2\sqrt{X+N}} \right)_{N=0} = \frac{1}{2\sqrt{X}}$$

then:

$$Z \approx \sqrt{X+N} = \sqrt{X} + \frac{1}{2\sqrt{X}}\,N$$

and the SNR is:

$$SNR(Z) = \frac{P(\sqrt{X})}{P\left(\frac{N}{2\sqrt{X}} \right)} = \frac{4\,P^2(\sqrt{X})}{P(N)}$$

because N and X are independent. Considering SNR(Y)=P(X)/P(N) we obtain then:

$$NF(T) = \frac{SNR(Y)}{SNR(Z)} = \frac{P(X)}{4\,P^2(\sqrt{X})}$$

When X is uniformly distributed in [0…a], using the results of the Example 7:

$$NF(T) \approx \frac{P(X)}{4\,P^2(\sqrt{X})} = \frac{1}{4}\,\frac{a^2}{3}\,\frac{4}{a^2} = \frac{1}{3} \quad \rightarrow \quad NF(T)_{dB} = 10\log\frac{1}{3} = -\,4.77\text{ dB}$$

5. The Probability Integral Transformation

Definition 2: Probability Integral Transformation

Suppose that a random variable X has a continuous CDF F, and let Y = F(X). This transformation from X to Y is called the probability integral transformation.

Theorem 3

Let X have continuous CDF F(x) and define the random variable Y as Y = F(X). Then Y is uniformly distributed on (0,1), that is, $P(Y \le y) = y$, $0 < y < 1$.

Proof. For Y = F(X) we have, for $0 < y < 1$,

$$P(Y \le y) = P(F(X) \le y) = P(F^{-1}[F(X)] \le F^{-1}(y)) = P(X \le F^{-1}(y)) = F(F^{-1}(y)) = y.$$

At the endpoints we have $P(Y \le y) = 1$ for $y \ge 1$ and $P(Y \le y) = 0$ for $y \le 0$, showing that Y has a uniform distribution.

The reasoning behind the equality:

$$P(F^{-1}[F(X)] \le F^{-1}(y)) = P(X \le F^{-1}{}_X(y))$$

is somewhat subtle and deserves additional attention. If F is strictly increasing, then it is true that $F^{-1}(F(x)) = x$. However, if F is flat, it may be that $F^{-1}(F(x)) \ne x$. Then $F^{-1}(F(x)) = x_1$, since $P(X \le x) = P(X \le x1)$ for any x in [x1, x2]. The flat CDF denotes a region of 0 probability $P(x_1 < X \le x) = F(x) - F(x_1) = 0$.
End Proof.

Example 8: Generating random variables according to a given CDF.
One very common use of the Theorem 3 is the generation of random numbers. Typically a programming language will produce random variables that are uniform on the interval (0,1). If you want a random variable distributed according to some CDF $G_Y(y)$, you may use the probability integral transformation to generate values from G_Y. As an example, let:

$$g_Y(y) = \lambda\, e^{-\lambda y}\, I(0,\infty)(y)$$

for some number $\lambda > 0$ (this is an exponential distribution). The CDF of Y is:

$$G_Y(y) = \int_{-\infty}^{y} g_Y(t)\, dt = \int_{0}^{y} \lambda\, e^{-\lambda t}\, dt = 1 - e^{-\lambda y}$$

Therefore, if Y has this CDF, according to the Theorem 3 the random variable $Z = G_Y(y) = 1 - e^{-\lambda y}$ will be uniformly distributed. Inverting this function results in:

$$G^{-1}{}_Y(z) = \frac{-\ln(1-z)}{\lambda}$$

Therefore, if Z is uniformly distributed, the transformation $h(Z) = (-\ln(1-z)) / \lambda$ will result in an exponentially distributed random variable.